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The Growth Rate Distribution of Firms: A Dynamic Model

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ABSTRACT

The paper introduces a dynamic model for firm growth, demonstrating that Gibrat's law is associated with a Laplace distribution of growth rates. By considering the relationship between size and growth rate, the analytical model predicts heavier tails than those observed in the Laplace distribution, indicating that Gibrat's law is not generally applicable. The theory is validated through an analysis of companies in the pharmaceutical sector, showing strong alignment with empirical data without the need for free parameters.

Keywords: Growth Rate Distribution, Firm Size, Firm Growth, Gibrat's Law, Laplace Distribution

JEL Classifications: C02, C22, C46, L11, O4, D21

1. INTRODUCTION

The evolution of firms is a topic of significant economic interest. To account for the distribution of firm sizes, Gibrat proposed that firm size *S(t)* and growth rates *g(t)* are statistically independent, a principle known as Gibrat's law or the law of proportionate effects (Gibrat, 1931). Typically, the size of a firm is measured by its total assets or sales (e.g. Stanley et al., 1996; Geroski, 2000; Bottazzi and Secchi, 2006; Kang, 2021). The growth process of the *i-th* firm can be described as:

$$\frac{dS_i(t)}{dt} = g_i(t)\, S_i(t) \tag{1}$$

At time step:

$$t' = t + \Delta t \tag{2}$$

the growth rate of the *i-th* firm is defined for a time interval $\Delta t = 1$ by:

$$g_i(t) = \ln\left(\frac{S_i(t')}{S_i(t)}\right) \tag{3}$$

According to this view, firm size results from independent multiplicative fluctuations (or shocks). The central limit theorem suggests that the logarithm of the firm size distribution converges to a Gaussian distribution, consistent with empirical observations (e.g. Fabritiis et al., 2003; Cabral and Mata, 2003; Reichstein and Jensen, 2005). The size distribution of firms can be given by a lognormal distribution of the form:

$$P_S(S) = \frac{1}{\sqrt{2\pi}\, S\sigma_{LN}}\ exp\left(-\frac{(\ln(S)-\mu)^2}{2\sigma_{LN}^2}\right) \tag{4}$$

with market specific scaling parameters μ and σ_{LN}.

However, empirical studies have shown that the anticipated Gaussian growth rate distribution is not backed by the data. Research indicates that the growth rate distributions of firms closely resemble a Laplace distribution, which is sharply peaked at the centre and features fatter tails compared to a Gaussian distribution. (see e.g. Stanley et al. (1996), Amaral et al. (1997), Bottazzi et al. (2002), Bottazzi and Secchi (2003), Bottazzi and Secchi (2005),Reichstein and Jensen (2005), Dosi et al. (2010), Alfarano et al.(2012), Coad (2012), Zou (2019)). Several explanations for this observation have been proposed (e.g. Fu et al. (2005), Kaldasch (2012), Metzig and

Gordon (2014), Ishikava et al.(2016), Bottazzi and Secchi (2019)).

Research further found that smaller firms tend to make more significant leaps in growth, as highlighted by several authors (e.g., Bottazzi and Secchi, 2006; Riccabonia et al., 2008; Gabaix, 2011; Aratat, 2019). The empirical analysis suggests a relationship between size and variance of growth rates. The standard deviation of growth rates can be written as:

$$std(g) = C_1 S^{\beta} \tag{5}$$

while C_1 is a constant and the exponent varies between $-0.15 < \beta < -0.2$.

In this paper a dynamic approach to the growth rate dynamics of firms is established. It is based on the growth rate dependent conservation equation of firms of a market. Derived is the stationary growth rate distribution $P(g)$ for an ensemble of firms with total number $\tilde{N}$.[1] It is based on the following assumptions:

(i) The growth rates of firms change through small random fluctuations on the growth rate scale.
(ii) New firms enter the market with an initial growth rate of $g = 0$.
(iii) Firms exit the market due to reasons such as mergers or bankruptcies at an exit rate $d(t,g)$, which is independent of the growth rate and proportional to the current number of firms $N(t,g)$.
(iv) During the specified time interval of investigation ΔT, the total number of firms $\tilde{N}$ can be considered approximately constant.[2]
(v) The size-variance relationship of growth rates (5) applies.

After presenting the theory, the model is compared to an empirical study by Fabritiis et al. (2003), which supports the proposed model.

2. THE THEORY

Considered is a market for a time interval ΔT. $N(t,g)$ is the number of firms falling in the growth rate interval g and $g+dg$, The growth rate distribution $P(t,g)$ at time step t is defined by:

$$P(t,g) = \frac{N(t,g)}{\tilde{N}(t)} \tag{6}$$

With the total number of firms at t:

$$\tilde{N}(t) = \int_{-\infty}^{\infty} N(t,g)\,dg \tag{7}$$

The dynamics of the ensemble of firms $N(t,g)$ is governed by the following conservation equation:

$$\frac{\partial N(t,g)}{\partial t} = r(t,g) - d(t,g) - \frac{\partial j(t,g)}{\partial g} \tag{8}$$

The time-dependent evolution of $N(t,g)$ is influenced by three processes, represented by the three terms on the right side of this equation. The number of firms increases at the rate $r(t,g)$ and decreases at the exit rate $d(t,g)$. The last term in (8) accounts for the time evolution of firms' growth rates. Based on assumption (i) firms perform a random walk on the growth rate scale. The rate $j(t,g)$ can therefore be modeled as:

$$j(t,g) = -D\frac{\partial N(t,g)}{\partial g} \tag{9}$$

where the parameter $D > 0$ is treated as a constant[3].

The rate at which firms exit the market per unit time, $d(t,g)$, is proportional to their total number $N(t,g)$ and a rate of $1/\tau$, where τ represents the average lifespan of a firm (iii). It can be expressed as:

$$d(t,g) = \frac{1}{\tau} N(t,g) \tag{10}$$

Applying (7) and (8) the total number of firms is given by:

$$\frac{d\tilde{N}(t)}{dt} = \tilde{r}(t) - \tilde{d}(t) \tag{11}$$

with the total rates:

$$\tilde{r}(t) = \int_{-\infty}^{\infty} r(t,g)\,dg \tag{12}$$

And

$$\tilde{d}(t) = \int_{-\infty}^{\infty} d(t,g)\,dg \tag{13}$$

Inserting (10) into (13) the total exit rate becomes with (7):

$$\tilde{d}(t) = \frac{1}{\tau}\tilde{N}(t) \tag{14}$$

Assumption (iv) requires:

$$\frac{d\tilde{N}}{dt} \cong 0 \tag{15}$$

From (11) follows therefore:

$$\tilde{r}(t) \cong \tilde{d}(t) \tag{16}$$

Based on assumption (ii) new firms enter the market at growth rate $g = 0$. With (14) and (16) the rate $r(t,g)$ can thus be written as:

$$r(t,g) \cong \frac{1}{\tau}\tilde{N}\delta(g) \tag{17}$$

where the Dirac δ-function ensures that the entry of firms is located at $g = 0$.[4] The evolution of the number of firms (8) becomes with (9), (10) and (17):

1 A tilde over variables indicates total numbers.

2 Assumption (iv) applies if the total number of firms $\tilde{N}(t)$ is much larger than time dependent variations of their number $\delta\tilde{N}(t)$, $\tilde{N}(t) \gg \delta\tilde{N}(t)$.

3 D is equivalent to a diffusion coefficient in particle physics.

4 The Dirac delta function is a distribution of the form: $\delta(g-g')=\infty$ for $g=g'$ and 0 for $g\neq g'$,normalized to one.

$$\frac{\partial N(t,g)}{\partial t} = \frac{1}{\tau}\tilde{N}\delta(g) - \frac{1}{\tau}N(t,g) + D\frac{\partial^2 N(t,g)}{\partial g^2} \quad (18)$$

Scaling (18) by the total number of firms $\tilde{N}$ we obtain with (6) a partial differential equation for the evolution of the growth rate distribution $P_0(t,g)$, consistent with Gibrat's law. Multiplying the relation with τ leads to: [5]

$$\frac{\partial P_0(t,g)}{\partial t}\tau = \delta(g) - P_0(t,g) + k^2\frac{\partial^2 P_0(t,g)}{\partial g^2} \quad (19)$$

while

$$k^2 = \tau D \quad (20)$$

The time evolution of the growth rate distribution is examined in Appendix A. The stationary growth rate distribution is determined by a Laplacian of the form:

$$P_0(g) = \frac{1}{2k}\exp\left(-\frac{|g|}{k}\right) \quad (21)$$

where the standard deviation of the distribution is:

$$std(g) = \sqrt{2}k \quad (22)$$

However, according to assumption (v), the standard deviation of the growth rate is a function of the firm size defined by (5). Comparing the relations (5) and (22) indicates that the parameter k must be a function of the firm size S:

$$k(S) = C_2 S^{\beta} \quad (23)$$

while:

$$C_2 = \frac{C_1}{\sqrt{2}} \quad (24)$$

Taking this size-dependence of the growth rate (23) into account the stationary growth rate distribution can be obtained from the integral:

$$P(g) = \int_0^{\infty} P_0(g, k(S)) P_S(S)\, dS \quad (25)$$

where the firm size distribution $P_S(S)$ is given by (4). It is convenient to consider the firm size S on a logarithmic scale, such that:

$$s = ln(S) \quad (26)$$

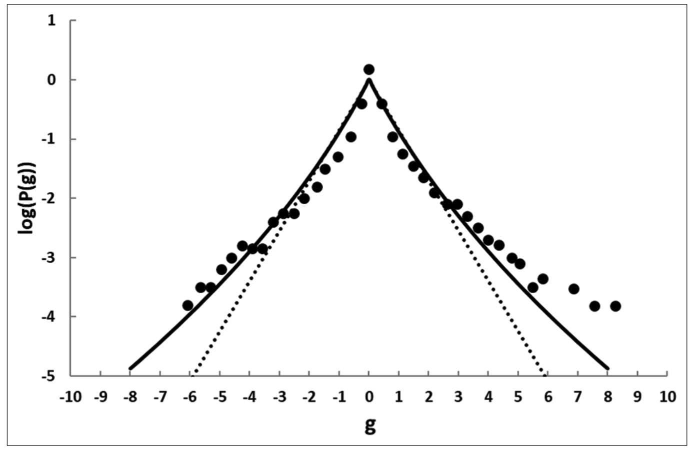
On this scale (4) becomes a normal distribution and (23) turns into:

$$k(s) = C_2 e^{\beta s} \quad (27)$$

The stationary growth rate distribution (25) has therefore finally the form:

5 Such a relation is known as a convection-diffusion equation.

Figure 1: Displayed is the empirical growth rate distribution of firms investigated by Fabritiis et al. (2003) (dots). The solid line is the numerical integration of (28) applying the reported data $\mu = 4.63$, $\sigma_N = 1.97$, $C_2 = 1.07$ and $\beta = -0.16$.[6] The dotted line indicates the growth rate distribution of firms of mean size μ.



$$P(g) = \int_{-\infty}^{\infty}\left(\frac{1}{2k(s)}\exp\left(-\frac{|g|}{k(s)}\right)\right)\left(\frac{1}{\sqrt{2\pi}\sigma_N} exp\left(-\frac{(s-\mu)^2}{2\sigma_N^2}\right)\right) ds \quad (28)$$

where μ is the mean firm size on a logarithmic scale, σ_N the corresponding standard deviation and $k(s)$ is given by (27).

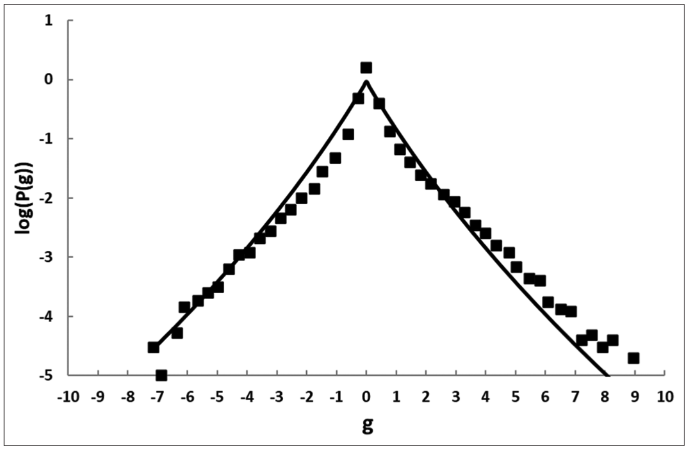
3. COMPARISON WITH AN EMPIRICAL INVESTIGATION

The model is applied to an empirical study conducted by Fabritiis et al. (2003), who extensively examined the size and growth distributions of products and business firms within the global pharmaceutical industry over a time span of $\Delta T = 10$ *years*. Utilizing data from the Pharmaceutical Industry Database (PHID) at CERM/EPRIS, they analyzed quarterly sales data for 48,819 pharmaceutical products marketed by 3,919 companies in the European Union and North America, covering the period from September 1991 to June 2001. Their findings indicated that the size distribution of firms aligns with a log-normal distribution, yielding the parameters $\mu = 4.63$ and $\sigma_N = 1.97$. Additionally, they observed that the standard deviation of firm growth rates tends to decrease with increasing firm size, as described by equation (5), with $C_1 \cong 1.51$ and $\beta \cong -0.16$. Figure 1 presents the empirical growth rate data from their research on a logarithmic scale (dots).

The theory presented relies on four parameters: μ, σ_N, C_2 and β. Since all these parameters are known for the market considered, the model can be applied without any free parameters. A numerical integration of (28) with (27) yields the solid line shown in Figure 1, demonstrating that the calculated growth rate distribution closely

6 For mathematical simplicity the presented model has neglected the Pareto tail of the size distribution. This contribution becomes relevant for small growth rates and may account for the discrepancies observed between the empirical data and the model near the centre of the distribution.

Figure 2: Displayed is the empirical growth rate distribution of products investigated by Fabritiis et al. (2003). The solid line is the numerical integration of (28) applying the empirical data $\mu = 4.34$, $\sigma_N = 1.59$, $C_2 = 1.12$ and $\beta = -0.16$.



aligns with empirical data.

The model allows an estimation of parameters for firms of average size of the market considered. With $\mu = 4.63$ the average value of the parameter k can be obtained from (27). For these firms the growth rate distribution is given by the Laplace distribution (21) with $k_\mu = 0.51$, displayed in Figure 1 by the dotted line. The mean lifetime τ can be obtained from (14). The investigators reported a nearly constant number of firms of the order $\tilde{N} \cong 2000$, which leads for an exit rate of firms of about $d \cong 200$ per year, to a mean lifetime $\tau \cong 10$ years. The relation (20) further yields $D_\mu \cong 2.63 \cong 10^{-2}$ per year.

4. CONCLUSION

Presented is a dynamic model for the evolution of firm growth rates, relying on a growth rate-dependent conservation equation of firms. According to this model firm's growth rate evolves by a small random amount over time. When accounting for firm entry and exit, the model suggests that Gibrat's law is associated with a Laplace distribution of the growth rates rather than a normal distribution (illustrated by the dotted line in Figure 1 for firms of average size). Smaller firms experience larger random fluctuations in their growth rates compared to larger firms, as indicated by Bottazzi and Secchi (2006) and Aratat (2019). Taking into account this size-variance relationship, the model predicts heavier tails than a standard Laplace distribution, a finding supported by empirical studies conducted e.g. by Reichstein and Jensen (2005), Buldyrev et al. (2007), and Bottazzi et al. (2011). Consequently, Gibrat's law is applicable only to firms of comparable size and does not hold true for a diverse range of firm sizes.

Applying the theory to firm growth in the pharmaceutical sector, as investigated by Fabritiis et al. (2003), a strong alignment with empirical data can be obtained when utilizing the reported data. Furthermore, Fabritiis et al. (2003) analysed the growth rate distribution of products, as shown in Figure 2 (squares). Assuming that the theory is applicable also to products, the model yields a growth rate distribution that aligns closely with the empirical data (solid line).[7] This finding indicates that the model may also be relevant for other economic growth rate data, provided that the model's conditions are met.

In summary, the established theory effectively captures the growth dynamics of firms and offers valuable insights into the relationship between growth and firm size, as supported by empirical evidence.

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7 Note that Fabritiis et al. (2003) reported that the number of products has increased linearly during ΔT, violating assumption (iv).

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APPENDIX A

The partial differential equation (19) can be solved by applying a one-dimensional Fourier transform of the distribution defined by:

$$F\left(P_0(t,g)\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} P_0(t,g)\exp\left(-i\omega g\right)dg \tag{A1}$$

and the inverse Fourier transform:

$$F^{-1}\left(P_0(t,\omega)\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} P_0\left(t,\omega\right)\exp\left(i\omega g\right)d\omega \tag{A2}$$

The Fourier transform of (19) with respect to g yields:

$$\frac{dP_0\left(t,\omega\right)}{dt}\tau = u(\omega)P_0\left(t,\omega\right) + \frac{1}{\sqrt{2\pi}} \tag{A3}$$

while

$$u(\omega) = -(1+k^2\omega^2) \tag{A4}$$

The time dependent ordinary differential equation (A3) can be solved with:

$$P_0\left(t,\omega\right) = C\left(\omega\right)\exp\left(u\left(\omega\right)\frac{t}{\tau}\right) - \frac{1}{\sqrt{2\pi}u\left(\omega\right)} \tag{A5}$$

where $C(\omega)$ is an integration constant. For $t\rightarrow\infty$, the stationary distribution becomes in Fourier space:

$$P_0\left(\omega\right) = -\frac{1}{\sqrt{2\pi}u\left(\omega\right)} \tag{A6}$$

The inverse Fourier transform of (A6) can be performed with (A2) using (A4):

$$F^{-1}\left[-\frac{1}{\sqrt{2\pi}u\left(\omega\right)}\right] = \frac{1}{2\sqrt{\tau D}}\exp\left(-\frac{|g|}{\sqrt{\tau D}}\right) \tag{A7}$$

Hence the stationary growth rate distribution has the form[8]:

$$P_0\left(g\right) = \frac{1}{2k}\exp\left(-\frac{|g|}{k}\right) \tag{A8}$$

8 The Fourier transform of (A5) yields the time evolution of (19). Starting with a normal distribution $P_0(t,g)$ approaches the Laplace distribution (A8) for $t>>\tau$.