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Tests of Parameters Instability: Theoretical Study and Empirical Analysis on Two Types of Models (ARMA Model and Market Model)

Sahbi FARHANI
Faculty of Economic Sciences and Management of Tunis, University of Tunis El Manar, Tunisia.
Tel: (+216)22833253. E-mail: sahbi_1984@yahoo.fr

Abstract: This paper considers tests of parameters instability and structural change with known, unknown or multiple breakpoints. The results apply to a wide class of parametric models that are suitable for estimation by strong rules for detecting the number of breaks in a time series. For that, we use Chow, CUSUM, CUSUM of squares, Wald, likelihood ratio and Lagrange multiplier tests. Each test implicitly uses an estimate of a change point. We conclude with an empirical analysis on two different models (ARMA model and simple linear regression model "SLRM").

Keywords: Tests of parameters instability; Structural change; Breakpoints; ARMA model; SLRM.
JEL Classifications: C22; Q43; G12

1. Introduction

The econometric analysis of time series is considered as an exclusive branch of econometrics. The latter is a relatively young discipline but the time series have been used before in many fields such as meteorology, astronomy, biology, economics, etc. The utility of time series consists to study variables over time and to examine the statistical analysis of observations regularly spaced in time. In fact, this study appears to have reached maturity during the 70s where significant developments have emerged. Indeed, conventional models couldn't fit the data if we take into account the circumstances that may disrupt these data sets and because that new tests have appeared such as tests of parameters instability and structural change.

Therefore, these tests consist to detect the existence of breakpoint(s) that may divide the total period into two or more sub-periods. After that, we pass to construct, for each sub-period, the appropriate model. But the problem proposed here, it is: how can we know that our model presents a change in its structure? And how can we detect the breakpoint(s) already existed?

In this context, it should be noted that the parameters instability study plays an important role when trying to understand the economic mechanisms and to make projections. This instability may reflect structural phenomena (model misspecification, omitted variables, measurement error, etc.) or punctual events (oil crisis, economic policy measures, new regulations, etc.).

In order to study the temporal instability problems, the econometricians have, recently, reached basic assumptions namely: the hypothesis of coefficients instability over time.

To resolve this problem, we start to study, in the second section, different tests of parameters instability (also called tests of structural change). After that, we passed to apply these tests on two different types of models.

The first model called ARMA, which will be applied to study the evolution of oil price. And the second one is rather a simple linear regression model used to study the evolution of financial asset prices following the change of market price.

The remainder of this paper proceeds as follows. Section 2 examines the theoretical tests of parameter instability and structural change. Empirical results are presented and discussed in Section 3. Conclusion is provided in the last section.

2. Tests of parameters instability and structural change: Theoretical study

In this section, we will study some tests to demonstrate the existence of parameters instability. These tests help us to find the best suitable model for such a time series that is to say that we must decide whether to keep the model in its integral form (if the parameters are fixed over the total period), or to divide it (if there is a breakpoint(s)). Several tests of parameters instability have been proposed in the literature with the most known: Chow test, likelihood ratio (LR) test, CUSUM test, CUSUM square test, Wald test, Lagrange multiplier (LM) test, etc.

To all tests listed above, there are some important limits as:

- Chow test aims to highlight the presence of a breakpoint at date "t", where this date should be known as of the beginning of study.
- CUSUM test consists to detect the instability of the intercept.
- Wald test allows detecting instability for both the intercept and coefficients.

2.1. Tests of parameters instability with known breakpoint: Chow (1960) Test

This test is most commonly used in time series analysis to reveal the existence of a structural breakpoint. It consists to divide the total period into two sub-periods (this is a simple case of the existence of one breakpoint). Indeed, we will consider the first case of the stability of model parameters over total period "T" (for null hypothesis, H_0), and we will assume the second case of the existence of a known breakpoint at the date T_1(for alternative hypothesis, H_1).

So we have two cases:

- The first one is: $y_t = \beta x_t + u_t$; $\forall t = 1,...T$ (1)
- The first one is: $y_t = \beta_i . x_t + u_{i,t} \begin{cases} i = 1 & \text{if } t = 1,...,T_1 \text{ and } T_1 > k \\ i = 2 & \text{if } t = T_1 + 1,...,T \text{ and } T - T_1 - 1 > k \end{cases}$ (2)

Where K is the number of parameters;

$u_{i,t}$ and u_t are independent and asymptotically distributed as normal distribution, N(0, σ^2).

In this case, the two hypotheses will be written as follows:

$$\begin{cases} H_0 : \beta_i = \beta & \text{; There is stability in parameters} \\ H_1 : \beta_i \neq \beta & \text{; There is \textbf{no} stability in parameters} \end{cases}$$

In this test, the detection of breakpoint amounts to detect a significant difference between the sum of squared residuals (RSS) in the case of stability coefficients and the sum of the two sum squared residual respectively associated to the first and second sub-period ($RSS_1 + RSS_2$).

$$RSS_1 = \sum_{t=1}^{T_1} \hat{u}_{1,t}^2 \tag{3}$$

$$RSS_2 = \sum_{t=T_1+1}^{T} \hat{u}_{2,t}^2 \tag{4}$$

$$RSS = \sum_{t=1}^{T} \hat{u}_t^2 \tag{5}$$

Chow test appears as a special case of Fisher (1970) test since it involves question about the equality between two coefficients groups. This test can be written in the form below:

$$\begin{cases} H_0 : R\beta = r \\ H_1 : R\beta \neq r \end{cases} \text{; Where } R = (I_{-k}, I_k) \text{ and } r = 0_{\mathbb{R}2k}$$

The statistical test is: $F = \frac{RSS - (RSS_1 + RSS_2)}{(RSS_1 + RSS_2)} \times \frac{T-2k}{k} \rightarrow F(k, T-2k)$ (6)

If F-statistic calculated exceeds F (k, T-2k), then we reject the null hypothesis of the absence of breakpoint and we will divide the total period into two sub-periods farthing this point (this date).

2.2. Tests of parameters instability with unknown breakpoint

Another approach proposes to use tests based on procedures that do not require necessarily knowledge of breakpoint date. Quandt (1960) discusses the null hypothesis of stable coefficients against a more general alternative hypothesis which assumes the existence of an unknown breakpoint. He considered a switching regression where the observations are divided into two separate regimes.

- The first regime is: $Y_t = X_t \beta_1 + \varepsilon_{1,t}$; $\forall t = 1,...,m$ (7)

- The second regime is: $Y_t = X_t\beta_2 + \varepsilon_{2,t} \quad ; \forall t = m+1,...,T$ (8)

We will study the flowing tests: likelihood ratio (LR), Wald, CUSUM and CUSUM of squares. These tests are based on least squares estimators.

2.2.1. Likelihood ratio (LR) test of Quandt (1960)

This test consists to verify if the observations in two consecutive time intervals come from the same regression or not. For that, we test:

i) H_0: The interval observations [1,m] and those of the interval [m+1,T] come from the same regression, ∀ m = k+1,…,T-k-1.

ii) H_1: The interval observations [1,m] and those of the interval [m+1,T] come from two different regressions.

Quandt (1960) proposes to estimate the date "m" by using the likelihood ratio. For this, he defined the following function:

$$\lambda_m = \log\left[\frac{\text{maximum likelihood} / H_0}{\text{maximum likelihood} / H_1}\right]$$

$$= \frac{m}{2}\log(RSS_1) + \frac{T-m}{2}\log(RSS_2) - \frac{T}{2}\log(RSS) \quad (9)$$

Where m = k +1,...,T-k-1, and k is the number of coefficients.

RSS_1, RSS_2, RSS are the sum of squared residuals sum of squares related to the number of observations when the regression is respectively calculated on the first "m" observations, on the following (T-m) observations, and on the total period ("T" observations).

If the likelihood ratio is high, then the probability that the alternative hypothesis H_1 is verified will be high, too.

By scanning the values of "m", we seek the minimum of the function that suggests a break in the tested regression. Otherwise the estimated point, at which the transition from one ratio to other occurs, is the value "m" which λ_m reaches its minimum.

The graphical analysis of λ_m should help us to determine whether the breakpoint is reached abruptly (random variation in regression coefficients) or is reached gradually (systematic variation in regression coefficients).

In the same line, Kim and Siegmund (1989) examined the likelihood ratio (LR) test to detect a structural change in a simple linear regression. They considered that the null hypothesis remains the same (H_0 : indicates no structural change) While the alternative hypothesis H_1 will divide into two parts:

$$\begin{cases} H_1^1 : \text{indicates that the change affects only the intercept.} \\ H_1^2 : \text{indicates that the change affects both the intercept and the slope.} \end{cases}$$

They tried to use these assumptions to determine the asymptotic distribution of likelihood ratio of Quandt (1960) by the statistic test under the null hypothesis of no structural change against two alternative hypotheses is functioned by Brownian movement (or Wiener process).

Recently, Deng and Perron (2008) tried to examine the limit distribution of the cusum of squares test under general mixing conditions.

2.2.2. CUSUM and CUSUM of squares tests

We consider a model with "K" coefficients varying over time:

$$Y_t = X_t\beta_t + \varepsilon_t \quad ; \forall t = 1,...,T \quad (10)$$

The estimated coefficients can be found by the ordinary least squares (OLS) method based on the first "t" observations: $\hat{\beta}_t = (X_t^{'}X_t)^{-1}X_t^{'}Y_t \quad ; \forall t = K,...,T$.

Similarly, the final estimator $\hat{\beta}_t$ of β_t is identical to the OLS classical estimator. So, it can be calculated by using the recursive estimators as follows: $\hat{\beta}_{t-1} = (X_{t-1}^{'} X_{t-1})^{-1} X_{t-1}^{'} Y_{t-1} \quad ; \forall t-1 > K$.

Finally, the estimator $\hat{\beta}_t$ of β_t will be written as follows:

$$\hat{\beta}_t = \hat{\beta}_{t-1} + (X_{t-1}^{'} X_{t-1})^{-1} . x_t . \frac{y_t - x_t^{'} \hat{\beta}_{t-1}}{1 + \left[x_t^{'} (X_{t-1}^{'} X_{t-1})^{-1} x_t \right]} \tag{11}$$

Where, $x_t = (x_{1,t}, ..., x_{K,t})$.

Brown, Durbin and Evans (1975) proposed coefficients stability tests based on recursive residuals defined as the standardized one step prediction errors.
The one step prediction error will be written as follows:

$$e_t = y_t - x_t^{'} \hat{\beta}_{t-1} = x_t^{'} (\beta_t - \hat{\beta}_{t-1}) + \varepsilon_t \tag{12}$$

Where, $E(e_t) = 0,\ V(e_t) = \sigma_\varepsilon^2 \left[1 + x_t^{'} (X_{t-1}^{'} X_{t-1})^{-1} x_t \right] \sigma_e^2$ and the one step prediction errors are not correlated. In addition,

$$e_t = x_t^{'} (\beta_t - \hat{\beta}_{t-1}) + \varepsilon_t = \varepsilon_t - x_t^{'} (X_{t-1}^{'} X_{t-1})^{-1} \sum_{j=1}^{t-1} x_j \varepsilon_j \tag{13}$$

Where, $E(e_t\ e_m) = 0 \ ; \forall t = 1, ..., T$.

Thus, we can define recursive residuals "w_t", as the normalized prediction errors:

$$w_t = \frac{e_t \sigma_\varepsilon}{\sigma_e} = \frac{y_t - x_t^{'} \hat{\beta}_{t-1}}{\sqrt{1 + \left[x_t^{'} (X_{t-1}^{'} X_{t-1})^{-1} x_t \right]}} \tag{14}$$

From these recursive residuals, Brown, Durbin and Evans (1975) developed graphical tests in order to accept or to reject the null hypothesis: $H_0 : \beta_1 = \beta_2 = ... = \beta_T = \beta$.

Indeed, if β_t is constant until time t = m and differs after this date (t > m), then the recursive residuals "w_t" will have a null average until the date t = m and an average differs to zero for the next period ($E(w_t) = 0$ for t = 1,..., m and $E(w_t) \neq 0$ for t > m).

2.2.2.1. CUSUM Test

This test provides solutions for the alternative hypotheses containing the unknown breakpoint "m" from the following quantity: $W_t = \sum_{t=K+1}^{m} \frac{w_t}{\hat{\sigma}_w}$ (15)

Where, $\hat{\sigma}_w^2 = \frac{1}{T-K-1} \sum_{t=K+1}^{m} (w_t - \overline{w})^2$, $\overline{w} = \frac{1}{T-K} \sum_{t=K+1}^{m} w_t$, and m = K+1,…,T.

Under the null hypothesis, W_m must be inside the corridor $\left[-L_m, L_m \right]$. Where;

$$L_m = \frac{a(2m + T - 3K)}{\sqrt{T-K}} \tag{16}$$

With "a" is respectively equal to 1.143, 0.948 and 0.850 at the levels 1%, 5% and 10%. The null hypothesis will be rejected if the quantity W_m cut $-L_m$ or L_m . It means that if the coefficients are not stable over time, then there may be a disproportionate number of recursive residuals W_t with the same sign which requires W_m exiting out of the corridor.

If there is a negative slope (respectively positive), then the one step predicted values will be higher (respectively lower) than the observed values.

In general, we use the CUSUM test to detect any systematic eventual movements where the coefficients values reflecting a possible structural instability. If a breakpoint is found, then we will reject the specification chosen throughout the period.

2.2.2.2. CUSUM of squares test

If we want to detect random movements (those that do not necessarily come from a structural change in coefficients), Brown, Durbin, Evans (1975) suggest CUSUM of squares test. This test, which uses the sum of squared recursive residuals, is based on the graph of the following quantity:

$$s_m = \frac{\sum_{t=K+1}^{m} w_t^2}{\sum_{t=K+1}^{T} w_t^2} = \frac{S_m}{S_T} \tag{17}$$

Under the null hypothesis "H_0", the quantity "s_m" follows a Beta distribution with a mean equals to E (s_m) = [(m-K)/(T-K)] and is framed by the corridor ±C+[(m-K)/(T-K)] where "C" is the Kolmogorov-Smirnov statistic. If s_m comes out of the corridor at the date t = m, then there exists a random rupture reflecting the instability of the coefficients for this date.

In the same way, we notice that many extensions were made to develop specially this test by Ploberger, Krämer and Alt (1989). In other extension, Krämer, Ploberger and Alt (1988) studied the dependent variable models. Kao and Ross (1995) investigated the models with correlated perturbations over time. In the same line, Ploberger and Krämer (1990) extend the CUSUM test based on OLS residuals, and they showed that this test can be applied with OLS residuals and not only with recursive residuals. The inconvenience of CUSUM and CUSUM of square tests is that these tests having asymptotically a low level of coefficients instability but not at the entire vector of coefficients. To solve this problem, Ploberger, Krämer and Kontrus (1989) suggested that the parameters test should be based on the fluctuation test rather than the recursive residuals. A similar study was suggested by Sen (1980) for the case of simple regression model and by Ploberger (1983) for the case of fluctuation test. In the same way, Ploberger, Krämer and Kontrus (1989) considered models with varying parameters and suggested that the fluctuation test is based on the rejection of the null hypothesis of parameter stability tests any time that these tests float a lot.

Their test statistic is:

$$S^{(T)} = \max_{t=K,\ldots,T} \frac{t}{\hat{\sigma}_T} \left\| (X^{(T)'} X^{(T)})^{1/2} (\hat{\beta}_i^{(T)} - \hat{\beta}_i^{(T)}) \right\|_\infty \tag{18}$$

Where;

$$\hat{\sigma}_T^2 = \frac{\sum_{t=K}^{T} (y_t - x_t' \hat{\beta}^{(T)})^2}{T-K} \tag{19}$$

And $\|.\|_\infty$ denotes the maximum norm.

They derived the limits of the distribution of the statistical test and they tabulated tested values by using Monte Carlo methods. They also showed that the fluctuation test has non-trivial local power regardless of the particular type of structural change. However, Ploberger, Krämer and Kontrus (1989) showed from Monte Carlo methods results that neither the fluctuation test nor the CUSUM test dominates each other for the small sample size. These authors compared both tests (otherwise the fluctuation test and the CUSUM test) and they found that the fluctuation test can help us in case of several points of breakpoints. Another type of tests was proposed by Leybourne and McCabe (1989), Nabeya and Tanaka (1988) and Nyblom (1989) whose considered the nonstationarity of parameters at the alternative hypothesis.

2.2.3. Wald test

The likelihood ratio (LR) test of Quandt (1960) and the procedures of Kim and Siegmund (1989) follow the same procedures of sup F test (also noted Max Chow) because their procedures seek to discover the meaning of the maximum value of the likelihood ratio statistic in models based on recursive residuals.

Andrews (1993) derived the asymptotic distribution for a similar test of likelihood ratio with a single unknown breakpoint (Quandt test). This is analogous to Wald (W) and Lagrange multiplier (LM) tests. This author proved that his "Sup F" test (or Max Chow) has a better power properties than CUSUM test and fluctuation test (especially in the case of linear model). He provided asymptotic critical values of significance levels with 1%, 2.5%, 5% and 10%. And he identified three regimes from autoregressive model of order 1. AR (1) is defined as follows:

$$y_t = \rho y_{t-1} + \varepsilon_t \ ; \forall t = 1,...,T \text{ , and } \varepsilon_t \to iid \ N(0,1) \quad (20)$$

Under the null hypothesis of absence of breakpoint, **Equation 20** will be estimated by OLS.

Assuming now that the model contains a breakpoint at time t = m, we will have therefore:

$$\begin{cases} y_t = \rho_1 y_{t-1} + \varepsilon_{1,t} \ ; \forall t = 1,...,m \\ y_t = \rho_2 y_{t-1} + \varepsilon_{2,t} \ ; \forall t = m+1,...,T \end{cases}$$

Where, $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$ are the residuals obtained, respectively, for sample 1 and sample 2.

With $RSS = \hat{\varepsilon}'\hat{\varepsilon}$, $RSS_1 = \hat{\varepsilon}_1'\hat{\varepsilon}_1$, and $RSS_2 = \hat{\varepsilon}_2'\hat{\varepsilon}_2$.

Sup-statistics suggested by Andrews (1993):

$$SupW = \max_{\pi} T.\left[\frac{RSS - (RSS_1 + RSS_2)}{(RSS_1 + RSS_2)}\right] \quad (21)$$

$$SupLM = \max_{\pi} T.\left[\frac{RSS - (RSS_1 + RSS_2)}{RSS}\right] \quad (22)$$

$$SupLR = \max_{\pi} T.\left[\frac{RSS}{(RSS_1 + RSS_2)}\right] \quad (23)$$

Where π = (m/T), it is customary to take π ∈ [0.15, 0.85], so that the breaks, which appear in the extremities, will be eliminated. Diebold and Chen (1996) showed that the use of asymptotic critical values suggested by Andrews (1993) leads to distortions especially if the test is performed for finite sample size, then they suggested that the best procedure to obtain critical values for finite samples size is the simulation methods (such as Bootstrap method).

Andrews and Ploberger (1994) developed these tests with stronger optimality conveniences than those developed by Andrews (1993). These statistics are defined by the Wald test.

We just notice that the definitions for the LM and LR statistics are similar and the asymptotic distributions are the same for the statistical laws of W, LM and LR.

Let $T_1 < m < T_2$, with $[T_1, T_2]$ is the interval where the breakpoint "m" appeared.

We consider Wald's test-statistic "W^*" at the break time t = m:

$$SupW = \max_{T_1 < m < T_2} W^* \quad (24)$$

The statistical tests suggested by Andrews and Ploberger (1994) are:

$$\exp W = \log\left[\frac{1}{T_2 - T_1 + 1}\sum_{m=T_1}^{T_2} \exp\left(\frac{W^*}{2}\right)\right] \quad (25)$$

$$moyW = \overline{W} = \frac{1}{T_2 - T_1 + 1}\sum_{m=T_1}^{T_2} W^* \quad (26)$$

Based on large regular conditions, Andrews (1993), Andrews and Ploberger (1994) and Andrews, Lee and Ploberger (1996) showed that the asymptotic distributions of statistical tests are given by Wiener process. The statistical tests suggested in this case are:

$$SupW = \max_{\pi_1 < \tau < \pi_2} Q(\tau) \quad (27)$$

$$\exp W = \log\left[\frac{1}{\pi_2 - \pi_1}\int_{\pi_1}^{\pi_2} \exp\left(\frac{Q(\tau)}{2}\right).d\tau\right] \quad (28)$$

$$moyW = \overline{W} = \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} \exp Q(\tau).d\tau \tag{29}$$

Where, $\pi_1 = (T_1/T)$, $\pi_2 = (T_2/T)$, $Q(\tau) = \frac{\left[W(\tau) - \tau W(\tau)\right]' . \left[W(\tau) - \tau W(\tau)\right]}{\tau(1-\tau)}$ and "W" is a vector with order k of independent Brownian movement on [0, 1] that if "τ" is known, then "W" follows a chi-square with order k.

2.2.3.1. Procedure of Wald test

Consider the parametric model indexed by parameters β_t for t = 1, 2,..., T. The important hypothesis considered here is concerned the stability of parameters: $H_0 : \beta_t = \beta_0 \;\; ; \forall t \geq 1$

The existence of unknown breakpoint "m" appears only in the alternative hypothesis which is given by: $H_1 : \beta_t = \begin{cases} \beta_{1,t} & \text{if } t = 1,...,T\pi \\ \beta_{2,t} & \text{if } t = 1+T\pi,...,T \end{cases}$

The unknown breakpoint "m" can be written as:

$$m = T.\pi \Leftrightarrow \pi = (m/T) \tag{30}$$

Where π ∈ [0,1] and "T" is the sample size. When "π" is known, Wald test will be equivalent to Fisher (Chow test). Indeed in this study, the statistic form is considered as follows:

$$\underset{\pi \in \Pi}{Sup}\, W_t(\pi), \text{ with } \pi \in \Pi \subset \left[0,1\right] \tag{31}$$

Where, Π is a known restricted interval containing all possible breakpoints. When we haven't information about the existence and the number of breakpoint, the choice will return to use Chow test.

In general, Wald test doesn't have standard asymptotic distribution. Because that we consider the set of breakpoints noted Π is [0,1] which isn't desirable because the statistical test doesn't converge in probability. But it may have some asymptotic properties in the case of sample with a high size and a low significance level. These results are proved by Davies (1987). But, if Π ⊂]0,1[, we will confirm that this statistic converges in distribution. So when no information is given about the breakpoint, we propose the use of restricted range Π= [0.15, 0.85].

2.2.3.2. Asymptotic properties of the Wald test statistic

2.2.3.2.1 Asymptotic distribution according to the null hypothesis

We will introduce the asymptotic distribution of Wald test statistic under the null hypothesis.

i) For set Π ⊂ [0,1], the following processes indexed by π ∈ Π satisfy:

$$W_t(\pi) => Q_p(\pi) \;\; ; \; SupW_t(\pi) \xrightarrow[d]{} SupQ_p(\pi)$$

$$Q_p(\pi) = \frac{\left[\left(\left(B_p(\pi) - \pi B_p(1)\right)' \left(B_p(\pi) - \pi B_p(1)\right)\right) - \pi B_p(1)\right]' . \left(B_p(\pi) - \pi B_p(1)\right)}{\pi(1-\pi)} \tag{32}$$

Where $B_p(\pi)$ is a vector of order "p" for independent movements in Brownian motions on [0,1] restricted to Π.

ii) For a fixed point $\pi \in [0,1]$, $Q_p(\pi)$ follows a chi-square distribution with "p" degrees. The asymptotic distribution of $SupW_t(\pi)$ under H_0 does not depend on nuisance parameters except the dimension "p". So, the critical values for statistics can be tabulated.

iii) The purpose of Π, to be far from 0 and 1, is taken to ensure that the estimators, described above on the statistical test which are presented, are uniformly consistent for $\pi \in \Pi$ and that the function $B_p(\pi)$ presented in $Q_p(\pi)$ is continuous.

iv) In fact, if $\Pi = [0,1]$ then the statistical test $SupW_t(\pi)$ will not converge in distribution.

2.2.3.2.2. Asymptotic critical values

The critical values of statistical test $SupW_t(\pi)$ are based on the asymptotic distribution, noted $SupQ_p(\pi)$. We propose "c_α" calculated by: Prob($SupQ_p(\pi) > c_\alpha$) = α.

If the interval "Π" is not symmetric, especially with Π = [π_1, π_2] and 0 < $\pi_1 < \pi_2$ < 1, then we will have:

$$\text{Prob}\left(SupQ_p(\pi) > c_\alpha\right) = \text{Prob}\Big(\sup_{s\in\left[1,\frac{\pi_2(1-\pi_1)}{\pi_1(1-\pi_2)}\right]} \frac{BM(s)^{'}BM(s)}{s} > c_\alpha\Big) \quad (33)$$

Where, BM(s) denotes a vector of order "p" with independent movement of Brownian motions on [0,+∞[.

Consequently, the critical values, based on the distribution $\sup_{\pi\in[\pi_1,\pi_2]} Q_p(\pi)$, depend to π_1 and π_2 only through the parameter "λ" defined as:

$$\lambda = \frac{\pi_2(1-\pi_1)}{\pi_1(1-\pi_2)} \quad (34)$$

We can calculate the critical values non-tabulated by a linear interpolation method.

2.3. Tests of parameters instability with multiple breakpoints

Consider the multiple linear regressions model with "m" breaks (that is to say that we have "m+1" sub-periods) below:

$$y_t = x_t^{'}\beta + z_t^{'}\delta_j + u_t \;\; ; \forall t = T_{j-1}+1,...,T_j \quad (35)$$

With j = 1,..., m+1, $T_0 = 0$ and $T_{m+1} = T$.

In this model (**Equation 35**), we can define y_t as a dependent variable observed at the time t, $x_t \in \mathbb{R}^p$ and $z_t \in \mathbb{R}^q$ are two vectors of regression, β and δ_j are two vectors of regression coefficients and u_t as an error term.

We consider the unknown breakpoints (T_1,..., T_m) with T_i=[λ_i T] for 0< $\lambda_1 < ... < \lambda_i < .. < \lambda_m$ <1.

If the date of breakpoint "m" is known, then (T_1,..., T_m, β, δ_1, δ_2,...,δ_{m-1}) will be estimated by the least squares method.

For any m-partition (T_1,..., T_m), denoted by $\{T_j\}$, the least squares estimators associated to β and δ_j are obtained by the minimizing the objective function $\sum_{i=1}^{m+1}\sum_{t=T_{i-1}+1}^{T_i}\left(y_t - x_t^{'}\beta - z_t^{'}\delta_i\right)^2$ under the constraint $\delta_i \neq \delta_{i+1}$; $\forall i = 1,...,m$.

We conclude after that the estimators $\hat{\beta}\{T_j\}$ and $\hat{\delta}\{T_j\}$ obtained by the sum of squared residuals $S_T\left(\mathrm{T}_1,...,\mathrm{T}_\mathrm{m}\right)$.

Finally, the estimators of $\left(\mathrm{T}_1,...,\mathrm{T}_\mathrm{m}\right)$ breakpoints are given by:

$$\left(\hat{\mathrm{T}}_1,...,\hat{\mathrm{T}}_\mathrm{m}\right) = S_T\left(\mathrm{T}_1,...,\mathrm{T}_\mathrm{m}\right) \tag{36}$$

And obtained by the minimization objective function under the constraint $T_i - T_{i-1} \geq$ h $\geq$ q.

If $\{T_j\}$ is the optimal partition, then the regimes will be estimated by:

$$\hat{\beta} = \hat{\beta}\left(\{T_j\}\right);\ \hat{\delta} = \hat{\delta}\left(\{T_j\}\right)$$

In this paper, we will explore the tests proposed by Bai and Perron (1998), Bai (1999), Bai and Perron (2003), Bai and Perron (2006), Bai and Ng (2007), Kim and Perron (2009), Kejriwal and Perron (2010) and Chen, Gerlach and Liu (2011) and adapted to many breakpoints and we propose to investigate the following three hypothesis tests:

Hypothesis.1: H_0: absence of breakpoints **against** H_1: presence of fixed and known breakpoints:

Bai and Perron (1998) considered the *sup*F test for testing:

$$\begin{cases} H_0\text{: Absence of breakpoints} \\ H_1\text{: Presence of “k” known breakpoints} \end{cases}$$

The Fisher statistic is given by:

$$F\left(\lambda_1,...,\lambda_k,q\right) = \frac{1}{T}\times\frac{T-(K+1)q-p}{kp}\times\frac{\delta^{'}R^{'}\left(R\left(\overline{Z}^{'}M_X\overline{Z}\right)^{-1}R^{'}\right)R\delta}{RSS_k} \tag{37}$$

R is the conventional matrix, $\left(R\delta\right)^{'} = \left(\delta_1^{'} - \delta_2^{'},...,\delta_k^{'} - \delta_{k+1}^{'}\right)$, $M_X = I - X^{'}(X^{'}X)^{-1}X$, RSS_k: is the sum of squared residuals under the null hypothesis, $X = \left(x_1,...,x_T\right)^{'}$, $\delta = \left(\delta_1^{'},...,\delta_{k+1}^{'}\right)$, $Z = \left(Z_{T_{i-1}+1},...,Z_{T_i}\right)^{'}$ and $\overline{Z} = diag\left(Z_1,...,Z_{k+1}\right)$.

Hypothesis.2: H0: absence of breakpoints **against** H1: presence of fixed and **un**known breakpoints:

Bai and Perron (2006) considered the DmaxF test for testing:

$$\begin{cases} H_0\text{: Absence of breakpoints} \\ H_1\text{: Presence of “k” \textbf{un}known breakpoints} \end{cases}$$

After fixing an upper bound for “k”, denoted “M”, the statistic of Fisher “DmaxF” will be written as follows:

$$D\max F_t(M,a_1,...,a_M,q) = \max_{1<m<M} a_m \sup_{\lambda_1,...,\lambda_k \in \Lambda_\in} F(\lambda_1,...,\lambda_k,q) \tag{38}$$

At this level, we can define the following set for an arbitrary positive number “ε”, it is an adjustment parameter which imposes a minimum length “h”, so it will be defined as ε =(h/T) and helps us to obtain the quantity:

$$\Lambda_\in = \left\{\left(\lambda_1,...,\lambda_k\right);\ \left|\lambda_{i+1} - \lambda_i\right| \geq \varepsilon,\ \lambda_1 \geq \in,\ \lambda_m \leq 1-\varepsilon\right\} \tag{39}$$

Where a_m is a weight function, $\forall$ 1< m < M.

Hypothesis.3: H0: presence of k breakpoints **against** H1: presence of (k+1) breakpoints:

Bai and Perron (2003) considered the Wald test for testing:

$$\begin{cases} H_0\text{: Presence of “k” breakpoints} \\ H_1\text{: Presence of “k+1” breakpoints} \end{cases}$$

In this case, the Wald statistic will be written as follows:

$$\sup F_t\left(k+\frac{1}{k}\right)=\frac{\left\{S_T\left(\hat{T}_1,...,\hat{T}_k\right)-\min\limits_{1<i<k+1}\ \inf\limits_{\tau\in\Lambda_{i,\in}} S_T\left(\hat{T}_1,...,\hat{T}_{i-1},\hat{\tau},\hat{T}_i,...,\hat{T}_k\right)\right\}}{\hat{\sigma}^2} \qquad (40)$$

Where, $\Lambda_{i,\in}=\left\{\tau,\hat{T}_{i-1}+\in.(\hat{T}_i-\hat{T}_{i-1})\le\tau\le\hat{T}_i+\in.(\hat{T}_i-\hat{T}_{i-1})\right\}$ and $\hat{\sigma}^2$ is the estimator of σ^2.

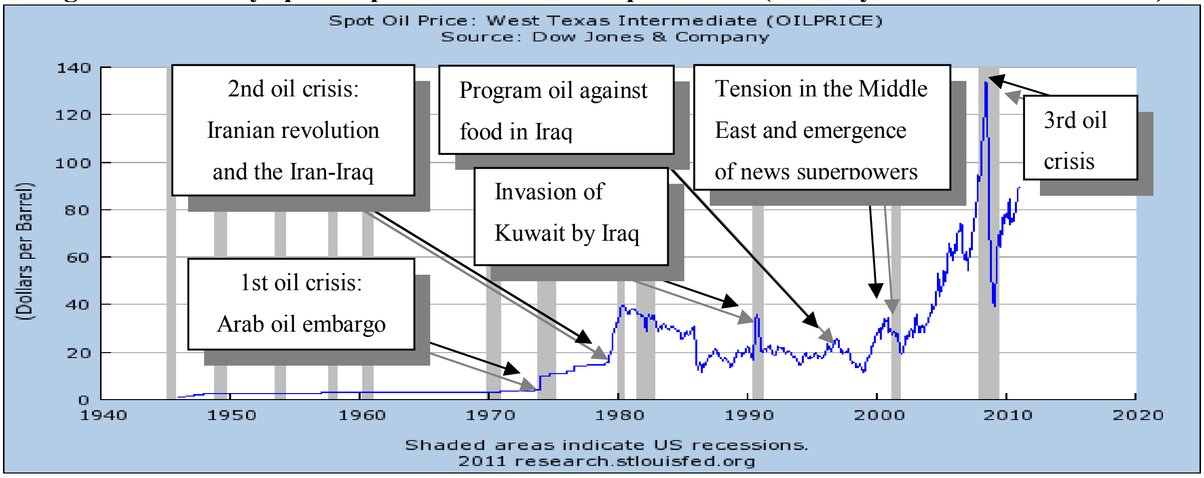
3. Tests of parameters instability and structural change: Empirical analysis

3.1. ARMA Model: Econometric approach based on U.S oil price

3.1.1. Statistical properties of data

Oil price data that we used in our study are monthly and measured in U.S. dollar per Barrel from January 1946 until December 2010[1]. Then we have 780 observations (presented in **Figure 1**).

Figure 1. Monthly spot oil price in U.S. dollars per barrel (January1946 – December 2010)



Firstly, we will study the stationarity of our series. Secondly, we will choose the appropriate ARMA model based on the optimal lags number. And finally, we will apply the parameter instability tests on the appropriate model to verify the presence of breakpoints. We will consider in this paper some details existed in the work of Chevallier (2011).

3.1.1.1. Unit root test (ADF test)

The first tests of Fuller (1976) and Dickey and Fuller (1979) are based on the estimation of Autoregressive (AR) process. These tests are the most used because of their simplicity, but they also suffer from several critics. These critics have led to another unit root tests proposing by Phillips and Perron (PP, 1988) and Kwiatkowski, Phillips, Schmidt and Shin (KPSS, 1992).

Generally, we use the Augmented Dickey-Fuller (ADF, 1981) test which consists to study the logarithmic series, denoted "$Ly_t=\log(y_t)$", as follows:

H_0: Ly_t is non stationary **Against** H_1: Ly_t is stationary

We begin by estimating the general model with intercept and trend (**Equation 41**):

$$DLy_t=\alpha+\beta.t+\phi.Ly_{t-1}+\sum_{j=1}^{p}\phi_j.DLy_{t-j}+\varepsilon_t \qquad (41)$$

Where DLy_t is Ly_t in first difference and "α" and "β" are, respectively, indicated intercept and trend coefficients.

The estimation of ADF models requires choosing the optimal lags number "p" concerned Autoregressive part. For this, we can use the correlogram of the series DLy_t finding in (**Figure 7**) to conclude that p = 1.

i) Estimate model with intercept and trend (**Equation 41**):

This is the same equation then we have: $DLy_t=\alpha+\beta.t+\phi.Ly_{t-1}+\sum_{j=1}^{p}\phi_j.DLy_{t-j}+\varepsilon_t$

[1] Source: http://research.stlouisfed.org/fred2/series/OILPRICE

According to the table (**Table 1**), the trend is not significantly different to zero (2.36 < 2.78), we pass then to estimate a model with intercept, but without trend.

ii) Estimate model with intercept, without trend (**Equation 42**):

$$DLy_t = \alpha + \phi.Ly_{t-1} + \sum_{j=1}^{p} \phi_j.DLy_{t-j} + \varepsilon_t \quad (42)$$

The intercept is not significantly different to zero (1.69 < 2.52) (**Table 1**), we pass then to estimate a model without intercept or trend.

iii) Estimate model without intercept and trend (**Equation 43**):

$$DLy_t = \phi.Ly_{t-1} + \sum_{j=1}^{p} \phi_j.DLy_{t-j} + \varepsilon_t \quad (43)$$

The statistical value given by ADF test is equal to 1.19 (**Table 1**), while it is higher than the theoretical value -1.95 at the 5% level. We therefore accept the null hypothesis of non stationary of the series Ly_t.

To determine the order of integration of this series, we will repeat the same procedure on the first difference of the series Ly_t, denoted DLy_t.

The graph of the series in first difference, (**Figure 6**), suggests that DLy_t is stationary.

i') Estimate model with intercept and trend (**Equation 44**):

$$D^2Ly_t = \alpha + \beta.t + \phi.DLy_{t-1} + \sum_{j=1}^{p} \phi_j.D^2Ly_{t-j} + \varepsilon_t \quad (44)$$

The trend is not significantly different to zero (0.03 < 2.78) (**Table 1**), we go then to estimate a model with intercept, but without trend.

ii') Estimate model with constant, without trend (**Equation 45**):

$$D^2Ly_t = \alpha + \phi.DLy_{t-1} + \sum_{j=1}^{p} \phi_j.D^2Ly_{t-j} + \varepsilon_t \quad (45)$$

The intercept is not significantly different to zero (1.84 < 2.52) (**Table 1**), we have then to estimate a model without constant or trend.

iii') Estimate model without intercept and trend (**Equation 46**):

$$D^2Ly_t = \phi.DLy_{t-1} + \sum_{j=1}^{p} \phi_j.D^2Ly_{t-j} + \varepsilon_t \quad (46)$$

The statistical value given by ADF test is equal to -22.1 (**Table 1**), and it is lower than the theoretical value of -1.95 at the level of 5%. We therefore reject the null hypothesis of non stationary of the series DLy_t.

We deduce that the series DLy_t is integrated of order zero, I (0). So, the series Ly_t is integrated of order one, I (1).

Table 1. Unit root test (ADF test) results of $Ly_t = LOIL\_PRICE_t$

Level			**1st difference**		
Trend & intercept	**Intercept**	**None**	**Trend & intercept**	**Intercept**	**None**
$H_0^3 : \phi = 0$	$H_0^2 : \phi = 0$	$H_0^3 : \phi = 0$	$H_0^3 : \phi = 0$	$H_0^2 : \phi = 0$	$H_0^3 : \phi = 0$
$t_{\hat{\phi}}$ =(-2,55)(> -3,54)	$t_{\hat{\phi}}$ =(-0,96)(> -2,95)	$t_{\hat{\phi}}$ = 1,19(> -1,95)	$t_{\hat{\phi}}$ =(-22,3)(< -3,54)	$t_{\hat{\phi}}$ =(-22,3)(> -2,95)	$t_{\hat{\phi}}$ =(-22,1)(< -1,95)
→ ϕ = 0	→ ϕ = 0	→ ϕ = 0	→ $\phi \neq 0$	→ ϕ = 0	→ $\phi \neq 0$
*Trend & intercept	*Intercept	*** Decision:**	*Trend & intercept	* Intercept	*** Decision:**
$t_{\hat{\delta}}$ = 2,36 (< 2,78)	$t_{\hat{c}}$ = 1,69 (< 2,52)	Ly_t **is NS**	$t_{\hat{\delta}}$ = 0,03 (< 2,78)	$t_{\hat{c}}$ = 1,84 (< 2,52)	DLy_t **is I(0)**
→ δ = 0	→ c = 0	**→1st difference**	→ δ = 0	→ c = 0	→ Ly_t **is I(1)**

NS: Non-Stationary

3.1.1.2. Figures analysis

According to the figures of the series $y_t = OIL\_PRICE_t$ (**Figure 1** and **Figure 2**), we note that it is not stationary. So, we can predict that there is a change in its structure following the first oil shocks of the global crises. This series appears non stationary (**Figure 2**) and this intuition can be enhanced by studying the correlogram (**Figure 3**) where all ACF (Autocorrelation Function) in this series are significantly different to zero and decreases slowly towards zero. So, we confirm the nonstationarity of this series.

Figure 2. Evolution of the series $y_t = OIL\_PRICE_t$

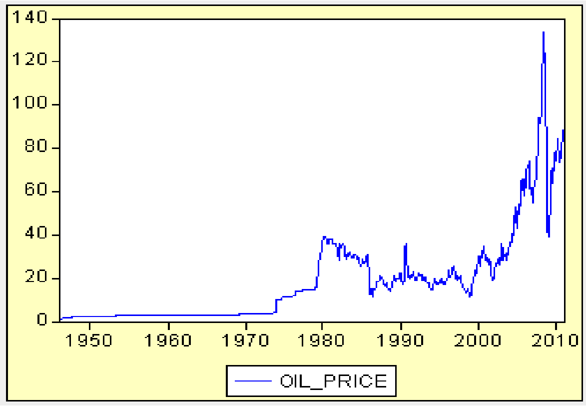
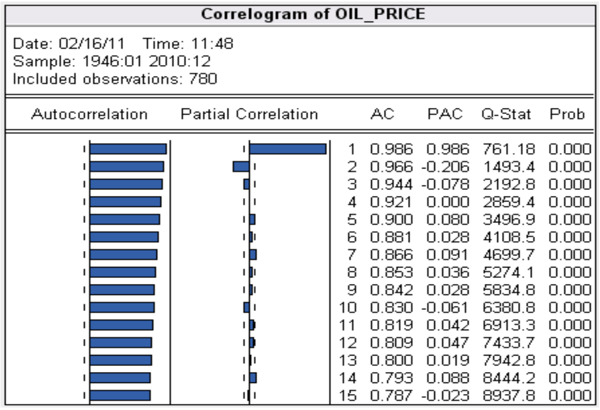


Figure 3. Correlogram of the series $y_t = OIL\_PRICE_t$



Correlogram of OIL_PRICE

Date: 02/16/11 Time: 11:48
Sample: 1946:01 2010:12
Included observations: 780

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.986	0.986	761.18	0.000
		2	0.966	-0.206	1493.4	0.000
		3	0.944	-0.078	2192.8	0.000
		4	0.921	0.000	2859.4	0.000
		5	0.900	0.080	3496.9	0.000
		6	0.881	0.028	4108.5	0.000
		7	0.866	0.091	4699.7	0.000
		8	0.853	0.036	5274.1	0.000
		9	0.842	0.028	5834.8	0.000
		10	0.830	-0.061	6380.8	0.000
		11	0.819	0.042	6913.3	0.000
		12	0.809	0.047	7433.7	0.000
		13	0.800	0.019	7942.8	0.000
		14	0.793	0.088	8444.2	0.000
		15	0.787	-0.023	8937.8	0.000

The including of logarithmic term to our series $y_t = OIL\_PRICE_t$, allows us to obtain $Ly_t = LOIL\_PRICE_t$, but it seems also non-stationary (**Figure 4**) and this intuition can be justified by studying the correlogram (**Figure 5**) where all ACF (Autocorrelation Function) in this series are significantly different to zero and decrease slowly towards zero. So, this result confirms the nonstationarity of this series, too.

Figure 4. Evolution of the series $Ly_t = LOIL\_PRICE_t$

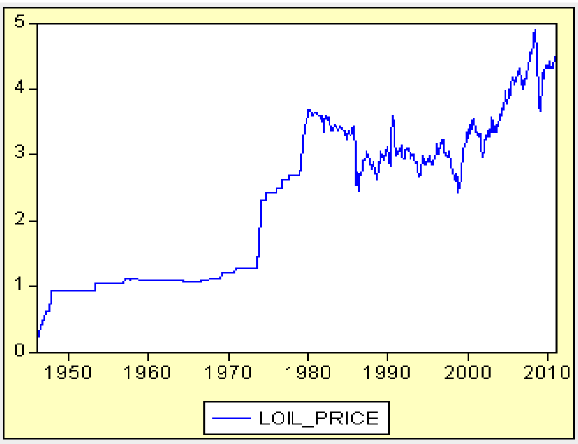
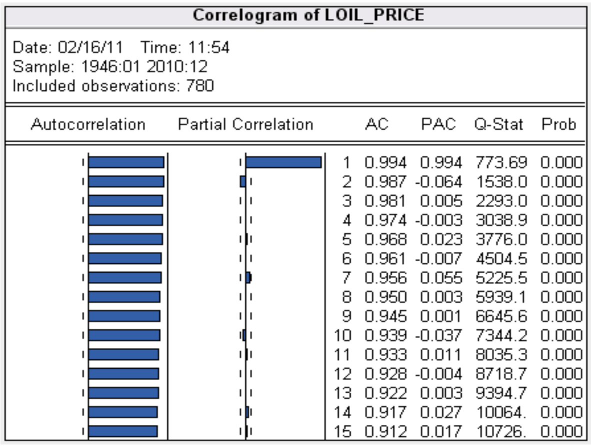


Figure 5. Correlogram of the series $Ly_t = LOIL\_PRICE_t$



Correlogram of LOIL_PRICE

Date: 02/16/11 Time: 11:54
Sample: 1946:01 2010:12
Included observations: 780

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.994	0.994	773.69	0.000
		2	0.987	-0.064	1538.0	0.000
		3	0.981	0.005	2293.0	0.000
		4	0.974	-0.003	3038.9	0.000
		5	0.968	0.023	3776.0	0.000
		6	0.961	-0.007	4504.5	0.000
		7	0.956	0.055	5225.5	0.000
		8	0.950	0.003	5939.1	0.000
		9	0.945	0.001	6645.6	0.000
		10	0.939	-0.037	7344.2	0.000
		11	0.933	0.011	8035.3	0.000
		12	0.928	-0.004	8718.7	0.000
		13	0.922	0.003	9394.7	0.000
		14	0.917	0.027	10064.	0.000
		15	0.912	0.017	10726.	0.000

But on the contrary, the series $DLy_t = DLOIL\_PRICE_t$ appears stationary (**Figure 6**) because the upward trend of the base series and even the logarithmic series was removed, and the average of the new series is around the x-axis. In addition, this result of the stationarity of this series can be justified by the correlogram (**Figure 7**) which don't show a particular structure at ACF (Autocorrelation Function) and PACF (Partial Autocorrelation Function).

Figure 6. Evolution of the series $DLy_t = DLOIL\_PRICE_t$

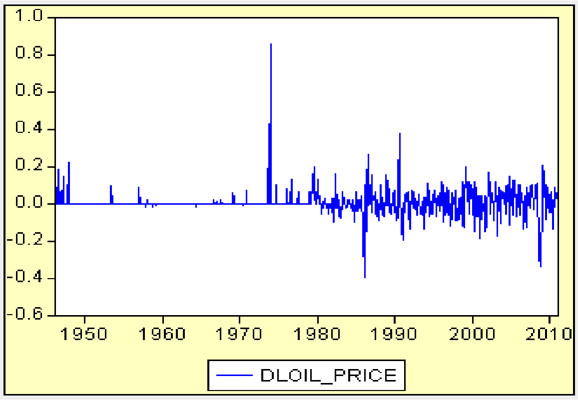
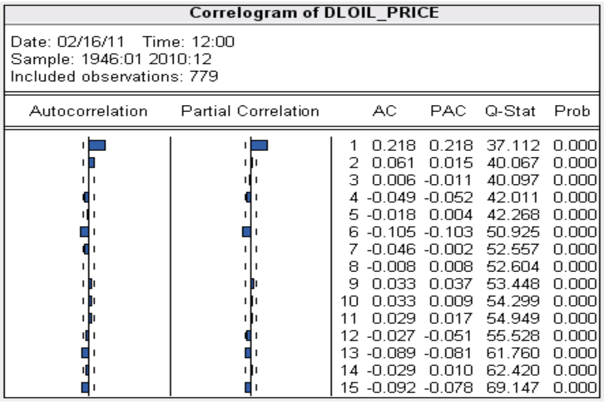


Figure 7. Correlogram of the series $DLy_t = DLOIL\_PRICE_t$


Correlogram of DLOIL_PRICE

Date: 02/16/11 Time: 12:00
Sample: 1946:01 2010:12
Included observations: 779

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.218	0.218	37.112	0.000
		2	0.061	0.015	40.067	0.000
		3	0.006	-0.011	40.097	0.000
		4	-0.049	-0.052	42.011	0.000
		5	-0.018	0.004	42.268	0.000
		6	-0.105	-0.103	50.925	0.000
		7	-0.046	-0.002	52.557	0.000
		8	-0.008	0.008	52.604	0.000
		9	0.033	0.037	53.448	0.000
		10	0.033	0.009	54.299	0.000
		11	0.029	0.017	54.949	0.000
		12	-0.027	-0.051	55.528	0.000
		13	-0.089	-0.081	61.760	0.000
		14	-0.029	0.010	62.420	0.000
		15	-0.092	-0.078	69.147	0.000

Series $DLy_t = DLOIL\_PRICE_t$ is stationary. So, the series $Ly_t = LOIL\_PRICE_t$ is stationary in first difference, then it is integrated of order 1, I(1). Therefore, DLy_t is I(0) and Ly_t is I(1). So, y_t is I(1).

3.1.1.3. Optimal lags number of ARMA (p, q) model

We begin by identifying "p" and "q" optimal lags with the use of the correlogram (**Figure 7**) of the stationary series, $DLy_t = DLOIL\_PRICE_t$. Looking at the same correlogram, we conclude that only the first Partial Autocorrelation (PAC) exceeds the confidence interval, so $\phi_{1,1} \neq 0$ → p = 1.

For completing the order "q", it's essential to watch the number of AC that exceeds the confidence interval. In this case, we find that only the two autocorrelation AC exceed the confidence interval, so $\varphi_1 \neq 0$ and $\varphi_2 \neq 0$ → q = 2.

Following these results, we obtained 3 process: AR(1), MA(2) and ARMA(1,2).

We will estimate these three models for choosing the most appropriate one:

* **<u>AR(1)</u>**: $DLy_t = c + \phi_1 DLy_{t-1} + \varepsilon_t$

$$\Leftrightarrow DLOIL\_PRICE_t = c + \phi_1 DLOIL\_PRICE_{t-1} + \varepsilon_t$$

$$\begin{cases} H_0: p' = p - 1 = 0 \Leftrightarrow \phi_1 = 0 \Leftrightarrow AR(0) \\ H_1: p' = p = 1 \Leftrightarrow \phi_1 \neq 0 \Leftrightarrow AR(1) \end{cases}$$

$t_{\hat{\phi}_1}$ = 6.217878 [> 1.96 (prob=0.0000)] (**Table 2**), then we will reject the null hypothesis and we will have $\phi_1 \neq 0$ → AR(1).

* **<u>ARMA (1,2)</u>**: $DLy_t = c + \phi_1 DLy_{t-1} + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2}$

$$\Leftrightarrow DLOIL\_PRICE_t = c + \phi_1 DLOIL\_PRICE_{t-1} + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2}$$

$$\begin{cases} H_0: p' = p - 1 = 0 \Leftrightarrow \phi_1 = 0 \Leftrightarrow MA(2) \\ H_1: p' = p = 1 \Leftrightarrow \phi_1 \neq 0 \Leftrightarrow ARMA(1,2) \end{cases}$$

$t_{\hat{\phi}_1}$ = 0.217428 [< 1.96 (prob= 0.8279 > 1%, 5% and 10%] (**Table 2**), then we will accept the null hypothesis and we will have $\phi_1 = 0$ → MA(2).

* **MA(2)**: $DLy_t = c + \varepsilon_t - \theta_1\varepsilon_{t-1} - \theta_2\varepsilon_{t-2}$

$\Leftrightarrow DLOIL\_PRICE_t = c + \varepsilon_t - \theta_1\varepsilon_{t-1} - \theta_2\varepsilon_{t-2}$

H_0: q' = q - 1 = 1 ⇔ $\theta_2 = 0$ ⇔ MA(1)

H_1: q' = q = 2 ⇔ $\theta_2 \neq 0$ ⇔ MA(2)

$t_{\hat{\theta}_2}$ = 1.720231 [< 1.96 (prob= 0.0858 > 5%)] (**Table 2**), then we will accept the null hypothesis and we will have $\theta_2 = 0$ → MA(1).

* **ARMA (1,2)**: $DLy_t = c + \phi_1 DLy_{t-1} + \varepsilon_t - \theta_1\varepsilon_{t-1} - \theta_2\varepsilon_{t-2}$

$\Leftrightarrow DLOIL\_PRICE_t = c + \phi_1 DLOIL\_PRICE_{t-1} + \varepsilon_t - \theta_1\varepsilon_{t-1} - \theta_2\varepsilon_{t-2}$

H_0: q' = q - 1 = 1 ⇔ $\theta_2 = 0$ ⇔ARMA(1,1)

H_1: q' = q = 2 ⇔ $\theta_2 \neq 0$ ⇔ ARMA(1,2)

$t_{\hat{\theta}_2}$ = 0.325451 [< 1.96 (prob=0.7449 > 1%, 5% et 10%)] (**Table 2**), then we will accept the null hypothesis and we will have $\theta_2 = 0$ → ARMA(1,1).

Until this stage, we obtain 4 processes AR (1), MA (1), MA (2) and ARMA (1,1).

Table 2. Estimation of ARMA(p, q) models (Dependent variable: DLOIL_PRICE)

Model	Variable	Coefficient	Std. Error	t-Statistic	Prob.
ARMA(1,0) → AR(1)	C	0,005575	0,003020	1,846340	0,0652
	AR(1)	0,217847	0,035036	6,217878	0,0000
ARMA(0,2) → MA(2)	C	0,005565	0,003006	1,850902	0,0646
	MA(1)	0,212526	0,035830	5,931549	0,0000
	MA(2)	0,061650	0,035838	1,720231	0,0858
ARMA(1,2)	C	0,005588	0,003044	1,835479	0,0668
	AR(1)	0,119559	0,549876	0,217428	0,8279
	MA(1)	0,094737	0,549622	0,172367	0,8632
	MA(2)	0,039003	0,119842	0,325451	0,7449

Subsequently, we test the absence of residuals autocorrelation:

* **AR (1):** the probabilities, presented in the figure (**Figure 8**), are all higher than 5%, then we deduce the absence of residuals autocorrelation. In this case, we conserve the model "AR(1)".

* **MA (1):** There is a presence of residuals autocorrelation in order of 6, 7, 13 and 15 verified by the figure (**Figure 9**). Then we will eliminate this model in the next stage.

Figure 8. Residuals Correlogram for AR(1)

Date: 02/20/11 Time: 19:43
Sample: 1946:03 2010:12
Included observations: 778
Q-statistic probabilities adjusted for 1 ARMA term(s)

	AC	PAC	Q-Stat	Prob
1	-0.003	-0.003	0.0073	
2	0.016	0.016	0.2141	0.644
3	0.004	0.004	0.2274	0.893
4	-0.052	-0.052	2.3160	0.509
5	0.016	0.015	2.5097	0.643
6	-0.101	-0.099	10.506	0.062
7	-0.024	-0.025	10.971	0.089
8	-0.006	-0.006	10.995	0.139
9	0.030	0.034	11.720	0.164
10	0.022	0.012	12.103	0.208
11	0.030	0.031	12.822	0.234
12	-0.016	-0.027	13.026	0.292
13	-0.084	-0.088	18.677	0.097
14	0.010	0.008	18.749	0.131
15	-0.074	-0.065	23.134	0.058

Figure 9. Residuals Correlogram for MA(1)

Date: 02/20/11 Time: 19:51
Sample: 1946:02 2010:12
Included observations: 779
Q-statistic probabilities adjusted for 1 ARMA term(s)

	AC	PAC	Q-Stat	Prob
1	0.011	0.011	0.0971	
2	0.058	0.058	2.7723	0.096
3	0.005	0.004	2.7938	0.247
4	-0.053	-0.057	5.0109	0.171
5	0.013	0.013	5.1355	0.274
6	-0.103	-0.098	13.510	0.019
7	-0.024	-0.023	13.953	0.030
8	-0.009	0.000	14.018	0.051
9	0.030	0.036	14.752	0.064
10	0.022	0.011	15.119	0.088
11	0.028	0.024	15.720	0.108
12	-0.016	-0.029	15.912	0.144
13	-0.087	-0.092	21.864	0.039
14	0.003	0.006	21.873	0.057
15	-0.078	-0.061	26.703	0.021

* **MA (2):** There is a presence of residuals autocorrelation in order of 6 and 15 verified by the figure (**Figure 10**). And we will eliminate this model in the next stage, too.

* **ARMA (1,1):** There is a presence of residuals autocorrelation to the order of 6 and 15 verified by the figure (**Figure 11**). Through to The next stage, this model will be eliminated.

Figure 10. Residuals Correlogram for MA(2)

Date: 02/20/11 Time: 20:02
Sample: 1946:02 2010:12
Included observations: 779
Q-statistic probabilities adjusted for 2 ARMA term(s)

	AC	PAC	Q-Stat	Prob
1	0.002	0.002	0.0018	
2	0.002	0.002	0.0048	
3	0.016	0.016	0.1977	0.657
4	-0.050	-0.050	2.1781	0.337
5	0.015	0.015	2.3477	0.503
6	-0.099	-0.100	10.120	0.038
7	-0.026	-0.024	10.651	0.059
8	-0.004	-0.007	10.661	0.099
9	0.029	0.034	11.308	0.126
10	0.021	0.012	11.665	0.167
11	0.031	0.032	12.433	0.190
12	-0.017	-0.029	12.674	0.242
13	-0.085	-0.088	18.376	0.073
14	0.011	0.009	18.470	0.102
15	-0.074	-0.066	22.798	0.044

Figure11.Residuals Correlogram for ARMA(1,1)

Date: 02/20/11 Time: 20:05
Sample: 1946:03 2010:12
Included observations: 778
Q-statistic probabilities adjusted for 2 ARMA term(s)

	AC	PAC	Q-Stat	Prob
1	0.000	0.000	3.E-06	
2	0.005	0.005	0.0167	
3	0.001	0.001	0.0172	0.896
4	-0.051	-0.051	2.0744	0.354
5	0.015	0.015	2.2578	0.521
6	-0.100	-0.100	10.182	0.037
7	-0.025	-0.025	10.679	0.058
8	-0.005	-0.007	10.696	0.098
9	0.031	0.033	11.441	0.120
10	0.023	0.013	11.857	0.158
11	0.031	0.032	12.615	0.181
12	-0.016	-0.027	12.827	0.234
13	-0.084	-0.087	18.391	0.073
14	0.010	0.009	18.471	0.102
15	-0.074	-0.066	22.792	0.044

=> Finally, We must retain the model: **AR(1)**.

3.1.2. Application of parameters instability tests

According to the figure of the series $DLy_t = DLOIL\_PRICE_t$ (**Figure 6**) and the result of the unit root test (ADF), we can conclude that this series is stationary, but we can predict the presence of breakpoints for some dates.

3.1.2.1. Tests of CUSUM, CUSUM of squares and recursive residuals

Applying the tests of CUSUM, CUSUM square and recursive residuals on the model AR(1), we will obtain these three figures (**Figure 12**, **Figure 13** and **Figure 14**).

Figure 12. CUSUM test

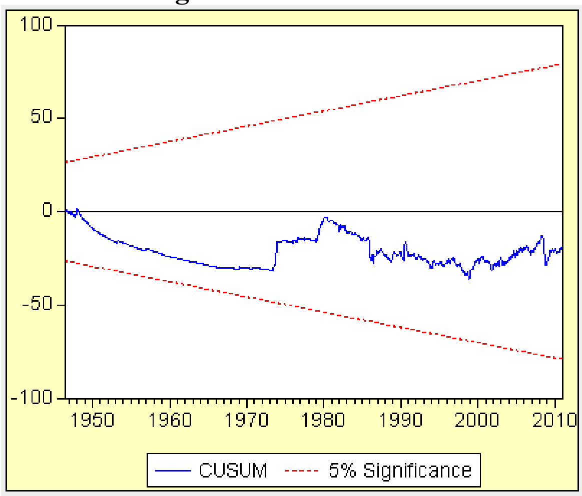


Figure 13. CUSUM of squares test

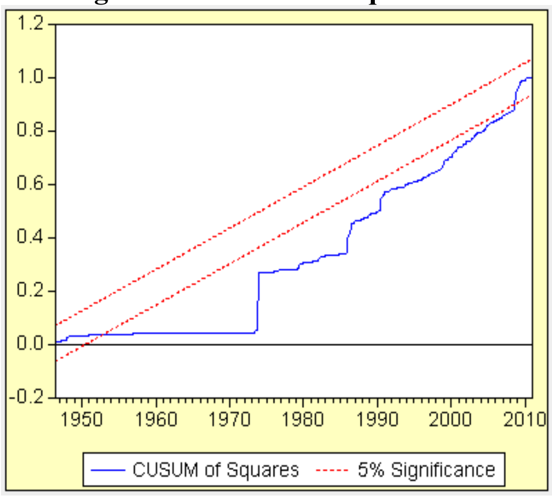
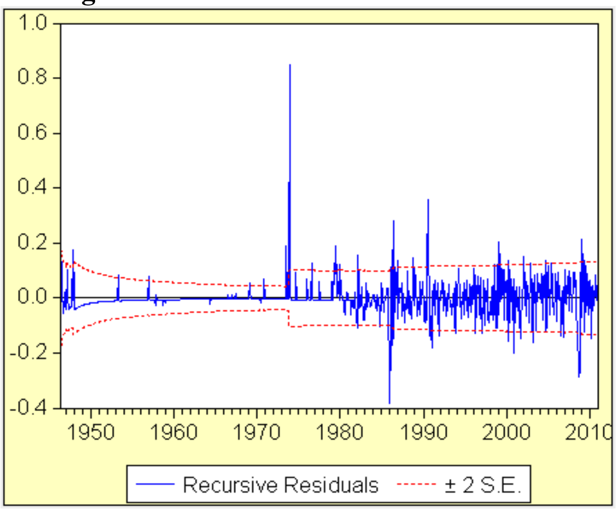


Figure 14. Recursive residuals test



For the CUSUM test (**Figure 12**), the test statistic is not outside the corridor. On the contrary, the statistic test of CUSUM of squares (**Figure 13**) is outside the corridor and it's the same for statistics of recursive residuals test (**Figure 14**). We conclude then the existence of break points.

3.1.2.2. Chow test

Any event doesn't cause a sudden shock, it is necessary to require the economic agents, the public authorities and the monetary authorities to take immediate decisions to reduce or even eliminate the effects of this shock on the economy. Hence the choice(s) item(s) will be chosen properly break from the figure (**Figure 1**) or (**Figure 2**).

By applying Chow test for the months of years 1973 and 1974, we can consider the month of February 1974 as a breakpoint because their p-value is equal to 0.000534 then it is less than 10%, 5% and even at the level 1% (See: **Table 3**).

Table 3. Chow test results (February 1974)

Chow Breakpoint Test:	**1974: 02**		
F-statistic	7,608498	Probability	0,000534
Log likelihood ratio	15,14727	Probability	0,000514

The estimation of the model AR (1) on the two new sub-periods gives us the **Table 4**.

Table 4. Estimation of AR(1) model (Dependent variable: DLOIL_PRICE)

Period	Variable	Coefficient	Std. Error	t-Statistic	Prob.
1946: 03 - 1974:01	C	0,006527	0,002859	2,282836	0,0231
	DLOIL_PRICE(-1)	-0,013965	0,054794	-0,254867	0,7990
1974: 02 - 2010:12	C	0,003464	0,003515	0,985242	0,3250
	DLOIL_PRICE(-1)	0,296694	0,045475	6,524376	0,0000

3.1.2.3. Results interpretation

We conclude that F-stat is very high (F-stat = 38.66200), the probability is null and the two coefficients are lower than the theoretical value "1.96" (**Table 5**). Hence we can predict the existence of one or many breakpoints.

Table 5. Fisher statistic and R^2 of AR(1) model (Dependent variable: DLOIL_PRICE)

Model	F-Statistic	Prob.	R^2
ARMA(1,0) → AR(1)	38,66200	0,000000	0,047458

If we divide the total period into two sub-periods as follows: (01/01/1946: 01/01/1974) and (02/01/1974: 12/31/2010), we note that in the first sub-period F-stat = 0.064957 is less than a Chi-square $\chi^2(p = 1)$ and the probability is higher than 1%, 5% and even 10%. But the R^2 is quite low (R^2=0,000195 → 0) (**Table 6**) and the coefficient linked to the series of oil price lagged by one period is not significant (= |-0.254867|<1.96) (**Table 4**) even with the elimination of the intercept. So there are another breakpoint(s) in this period, that it (they) can be predicted by the figure of recursive residuals (**Figure 14**).

As against for the second sub-period, we see that the coefficient associated to the series of oil price lagged by one period is significant (= 6.524376 > 1.96) (**Table 4**) and remains significant even with the elimination of the intercept, but F_stat = 42.56748 is higher than a Chi-square $\chi^2(p = 1)$ (**Table 6**) and the probability is null. So we can predict, with the use of the figure of recursive residuals (**Figure 14**), the existence of other breakpoint(s). Finally we conclude that with "p" breakpoints, we present (p +1) periods and that's return to estimate (p +1) models.

Table 6. Fisher statistic and R^2 of AR(1) model (Dependent variable: DLOIL_PRICE)

Period	F-Statistic	Prob.	R^2
1946: 03 - 1974:01	0,064957	0,798983	0,000195
1974: 02 - 2010:12	42,56748	0,000000	0,088028

3.2 Simple Linear regression model: Econometric approach based on the market model

With the market model, we can describe the specification of the return-generating process for the return of a financial asset "j" related directly with the market return. In other hand, the linear market model implies the security market line associated with the Capital Asset Pricing Model (CAPM) proved by Sharpe (1963, 1964), Lintner (1965), and Mossin (1966). We will start reminding the essential characteristics of this type of model. Subsequently, we consider what methods can be helped us to estimation a market model. We will examine in particular the stability of the relationship between these two factors.

3.2.1. Theoretical study of the market model

3.2.1.1. Model presentation

The market model was presented earlier by Sharpe (1963).

We notice by $R_{j,t}$, the return of the financial asset "j" measured at time "t":

$$R_{j,t} = \log(P_{j,t}) - \log(R_{j,t-1}) \quad (47)$$

Where, "$P_{j,t}$" is the price of the financial asset "j" measured at time "t".

We notice by "$R_{M,t}$" the market return.

The market model explains the evolution of financial asset returns "j" by the market return. The relationship between these two factors can be written as a linear regression model:

$$R_{j,t} = \alpha_j + \beta_j . R_{M,t} + \varepsilon_{j,t} \ ; \ \varepsilon_{j,t} \rightarrow N(0, \sigma_\varepsilon^2) \tag{48}$$

3.2.1.2. Model estimation

3.2.1.2.1. Prior assumptions

There are five conventional assumptions, such as:

$E(\varepsilon_{j,t}) = 0 \ ; \ V(\varepsilon_{j,t}) = \sigma_\varepsilon^2 \ ; \ E(\varepsilon_{j,t}\varepsilon_{j,t'}) = 0 \ ; \ E(\varepsilon_{j,t}R_{M,t}) = 0 \ ; \ E(\varepsilon_{j,t}\varepsilon_{k,t}) = 0$

3.2.1.2.2. Estimation of the coefficients

The two coefficients α_j and β_j can be estimated by OLS method. And the final result can be given by these two equations:

$$\hat{\beta}_j = \frac{\sum_{t=1}^{T} (R_{j,t} - \overline{R}_j)(R_{M,t} - \overline{R}_M)}{\sum_{t=1}^{T} (R_{M,t} - \overline{R}_M)^2} \tag{49}$$

$$\hat{\alpha}_j = \overline{R}_j - \hat{\beta}_j . \overline{R}_M \tag{50}$$

Where $\overline{R}_j$ and $\overline{R}_M$ are respectively the average of $R_{j,t}$ and $R_{M,t}$ for t = 1,2,...,T..

With estimated values of α and β, we deduce the calculated return of the financial asset "j":

$$\hat{R}_{j,t} = \hat{\alpha}_j + \hat{\beta}_j . R_{M,t} \tag{51}$$

This equation can be also written as follows:

$$R_{j,t} = \hat{R}_{j,t} + \hat{\varepsilon}_{j,t} = \hat{\alpha}_j + \hat{\beta}_j . R_{M,t} + \hat{\varepsilon}_{j,t} \tag{52}$$

3.2.1.2.3. Interpretation of the coefficients

i) The slope β_j measures the relative volatility of the financial asset return "j":

°If $\beta_j = 1$, that's mean that an increase in market return by 1%, increase the financial asset return by 1% too.

°If $\beta_j > 1$, that's mean that the assets "j" led to amplify market movement; (and they called offensive assets).

°If $\beta_j < 1$, that's mean that the assets "j" led to reduce market movement; (and they called defensive assets).

ii) The intercept can be interpreted as a value of the financial asset return when the coefficient of the marker return is null.

3.2.2. Statistical properties of data

In this section, we will study the financial asset return "TUNSIESICAV"[2] relative to the market return[3]. The data used are daily from January 3, 2003 until December 29, 2006, so we have 981observations.

Where "X" indicates the market return $R_{M,t}$ and "Y" indicates the financial asset return $R_{j,t}$ (TUNISIESICAV).

3.2.3. Application of parameter instability tests on the model market

According to the figure (**Figure 15**), we can see that the two series are stationary, but it is also possible that they represent breakpoints.

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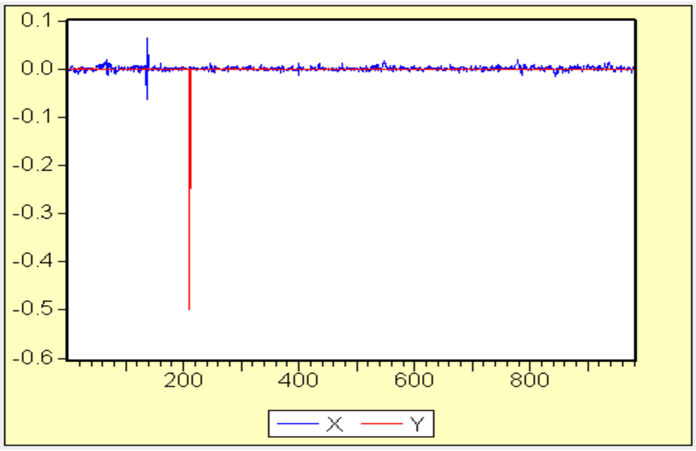
[2] Source: http://www.tunisievaleurs.com.tn or http://www.bloomberg.com/quote/TUNISIE:TU

[3] Source: http://www.bct.gov.tn/

Figure 15. Evolution of the returns $Y = R_{j,t}$ **and** $X = R_{M,t}$

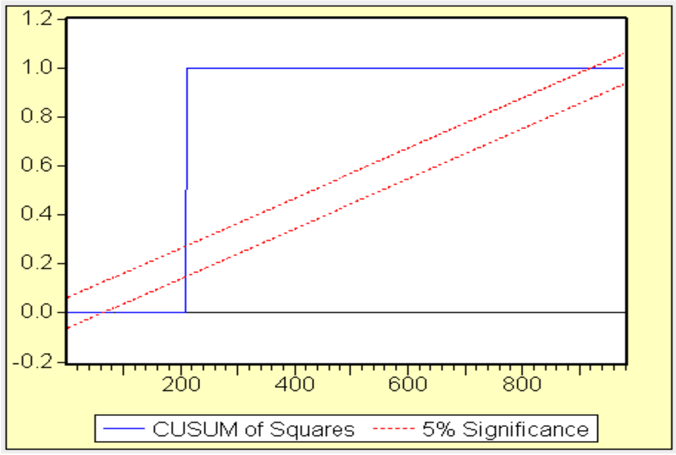


From the series "Y", which indicates the return of the financial asset "j", we can detect a breakpoint in the date January 12, 2003 (01/12/2003), and this result can be justified by the CUSUM and Chow tests indicated below.

3.2.3.1. CUSUM of squares test

The CUSUM of squares test statistic (**Figure 16**) comes out of the corridor and cut the margin of significance three times, so we conclude the existence of breakpoints supported by the Chow test indicated below.

Figure 16. CUSUM of squares test on market model $R_{j,t} = \alpha_j + \beta_j.R_{M,t} + \varepsilon_{j,t}$



3.2.3.2. Chow test

According to the **Equation 48**, we can detect a breakpoint on the date 01/12/2003 (observation 212) checked by the Chow test (1960) (**Table 7**).

P-value = 0.000002 is less than 10%, 5% and even 1%. That's mean that we can consider this date as a breakpoint.

Table 7. Chow test results

Chow Breakpoint Test:				
01/12/2003	F-statistic	13,17557	Probability	0,000002
(observation 212)	Log likelihood ratio	26,10759	Probability	0,000002

3.2.3.3. Results interpretation

When we estimate the **Equation 48**: $R_{j,t} = \alpha_j + \beta_j.R_{M,t} + \varepsilon_{j,t}$ at the total daily period daily from January 3, 2003 until December 29, 2006, we notice that F_stat value is very small (F-stat = 0.022175).

This result leads us to say that there are no breakpoints during this period, but from figures (**Figure 15**) and (**Figure 16**), we can note that the two series present breakpoints. For that, we must divide the total period to correct the perturbation series and contradictory results found. The first sub-period (1 to 211), we note that F-stat = 5.848133 > $\chi^2(1)$ (**Table 8**), so there is a breakpoint. On the remainder in (212 to 981), we note that F_stat = 0.036692 < $\chi^2(1)$ (**Table 8**), so there is not a breakpoint.

Table 8. Fisher statistic and R² of market model

Period	F-Statistic
01/03/2003- 01/11/2003 (1 to 211)	5,848133
01/12/2003- 12/29/2006 (212 to 981)	0,036692

4. Conclusion

Tests of parameter instability consist to detect the existence of breakpoints. In this case, we need to divide our total period into two or many sub-periods to improve the final results especially when the coefficients are not significant and the adjustment coefficient is low.

The advantage of these tests is that they can be applied to any model, and because that we have choose to applied it on the two different type of models such as ARMA model (by using the series of oil price) and linear regression model (by using the market model).

But our work can be developed by detection the existence of breakpoints in other series like insider trading (Olmo, Pilbeam and Pouliot, 2011) by changing the model and estimating for example structural changes in regression quantiles (Oka and Qu, 2011), etc.

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